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LAGRANGIAN FIELD THEORY AND SYMMETRIES FOR A BUNDLE $\pi: E \rightarrow \mathbb{R}^k$

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ABSTRACT. By generalizing the cosymplectic setting for time-dependent Lagrangian mechanics, we propose a geometric framework for the Lagrangian formulation of classical field theories with a Lagrangian depending on the independent variables. For that purpose we consider the first order jet bundles $J^1\pi$ of a fiber bundle $\pi: E \rightarrow \mathbb{R}^k$ where \mathbb{R}^k is the space of independent variables. Generalized symmetries of the Lagrangian are introduced and the corresponding Noether Theorem is proved.

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1. INTRODUCTION

As is well-know, the natural arena for studying mechanics is symplectic geometry. One interesting problem is to extend this geometric framework for the case of classical field theories. Several different geometric approaches are well known: the polysymplectic formalisms developed by Sardanashvily et al. [13, 14, 39], and by

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Kanatchikov [19], as well as the n -symplectic formalism of Norris [31, 33], and the k -cosymplectic of de Leon et al. [25, 26].

Let us remark that the multisymplectic formalism is the most ambitious program for developing the Classical Field Theory (see for example [2, 11, 12, 15, 16, 20], and references quoted therein).

The aims of this paper are

- To give a new Lagrangian description of first order Classical Field Theory, by considering a fibration $E \rightarrow \mathbb{R}^k$, which has as particular cases the cosymplectic setting for time-dependent Lagrangian mechanics, and the k -cosymplectic formalism [26].
- To introduce and to study the generalized symmetries in first order Lagrangian Field Theory. For time-dependent Lagrangian Mechanics this was done by J.F. Cariñena *et al.* [5].

In the present paper we present a new approach for Lagrangian Field Theory, working with the first order jet bundle $J^1\pi$ of a fiber bundle $\pi : E \rightarrow \mathbb{R}^k$, where E is $(n+k)$ -dimensional. The crucial point is that each 1-form on \mathbb{R}^k defines a tensor field of type $(1,1)$ on $J^1\pi$, see Saunders [41].

The paper is organized as follows. The main tools to be used are those of vector fields, k -vector fields and forms along maps, the general definitions of which are given in Section 2.

In Section 3 we introduce the geometric elements on $J^1\pi$ necessary to develop the geometric formulation of the Euler-Lagrange field equations in Section 4 and to the study of symmetries and conservations laws. The principal tools, here described, are the canonical vector fields, the k vertical endomorphisms and a kind of k -vector fields, known as SOPDES, which describe systems of second order partial differential equations.

The geometric formulation of the Euler-Lagrange field equations is given in Section 4, see Theorem 4.5. For this purpose we introduce the k Poincaré-Cartan 1-forms using the Lagrangian and the k vertical endomorphisms.

Our formulation is a natural extension of the k -cosymplectic formalism developed in [26] as we show in Section 4.3.

Section 5 is devoted to discussing symmetries and conservation laws. We introduce symmetries of the Lagrangian and we give a Noether Theorem.

2. PRELIMINARIES

2.1. k -vector fields. A system of first-order ordinary differential equations on a manifold M can be geometrically described as a vector field on M . Accordingly, a system of first-order partial differential equations on M can be geometrically described as a k -vector field on M , for some $k > 1$.

Particularly, we can identify a system of second-order partial differential equations (SOPDE) with some special k -vector fields on the manifold $J^1\pi$, for some $k > 1$.

We briefly recall the correspondence between systems of first-order partial differential equations and k -vector fields.

Let us denote by $T_k^1 M$ the Whitney sum $TM \oplus \dots \oplus TM$ of k copies of TM and $\tau_M : T_k^1 M \rightarrow M$ the canonical projection.

Definition 2.1. A k -vector field on an arbitrary manifold M is a section $\mathbf{X} : M \rightarrow T_k^1 M$ of the canonical projection $\tau_M : T_k^1 M \rightarrow M$.

Each k -vector field \mathbf{X} defines a family of k vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ by projecting \mathbf{X} onto every factor; that is, $X_\alpha = \tau_\alpha \circ \mathbf{X}$, where $\tau_\alpha : T_k^1 M \rightarrow TM$ is the canonical projection on the α^{th} -copy TM of $T_k^1 M$.

Definition 2.2. An integral section of the k -vector field $\mathbf{X} = (X_1, \dots, X_k)$, passing through a point $x \in M$, is a map $\psi : U \subset \mathbb{R}^k \rightarrow M$, defined on some open neighborhood U of $0 \in \mathbb{R}^k$, such that

$$\psi(0) = x, \quad \psi_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) = X_\alpha(\psi(x)) \in T_{\psi(x)} M, \quad (2.1)$$

for every $x \in U$, $1 \leq \alpha \leq k$.

A k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ on M is said to be integrable if there is an integral section of \mathbf{X} passing through every point of M .

From Definition 2.2 we deduce that ψ is an integral section of $\mathbf{X} = (X_1, \dots, X_k)$ if, and only if, ψ is a solution to the system of first-order partial differential equations

$$X_\alpha^i(\psi(x)) = \frac{\partial \psi^i}{\partial x^\alpha} \Big|_x, \quad 1 \leq \alpha \leq k, 1 \leq i \leq \dim M,$$

where $X_\alpha = X_\alpha^i \partial / \partial y^i$ on a coordinate system (U, y^i) on M , $y^i \circ \psi = \psi^i$, and x^α are coordinates on \mathbb{R}^k .

For a k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ on M we require the integrability condition $[X_\alpha, X_\beta] = 0, \forall \alpha, \beta \in \{1, \dots, k\}$, as has been considered in [28].

2.2. Sections along a map.

Given a fiber bundle $\pi : B \rightarrow M$ and a differentiable mapping $f : N \rightarrow M$, a section of π along f is a differentiable map $\sigma : N \rightarrow B$ such that $\pi \circ \sigma = f$ (see e. g. [36]). When π is a vector bundle, then the set of such sections can be endowed with a structure of $C^\infty(N)$ -module.

In the case of $\pi : B \rightarrow M$ being the tangent bundle of M , $\tau_M : TM \rightarrow M$, or the cotangent bundle, $\pi_M : T^*M \rightarrow M$, the sections along f will be called *vector fields along f* and *1-forms along f* , respectively.

The notion of sections along a map has been shown to be very fruitful. Many objects commonly used in Physics find their suitable geometric representative by means of this concept and a related one of f -derivations ([35, 41]), but they have only recently introduced in Physics [4, 6, 17].

Let X be a vector field along $f : N \rightarrow M$, say

$$\begin{array}{ccc} & & TM \\ & \nearrow X & \downarrow \tau \\ N & \xrightarrow{f} & M \end{array}$$

then we can define an f -derivation $i_X : \Lambda^p(M) \rightarrow \Lambda^{p-1}(N)$ of degree -1 (and type i_*) as follows: $i_X g = 0$ for all $g \in C^\infty(M)$ and

$$(i_X \omega)(x) (v_{1_x}, \dots, v_{(p-1)_x}) = \omega(f(x)) (X(x), f_*(x)v_{1_x}, \dots, f_*(x)v_{(p-1)_x}) \quad (2.2)$$

where $v_{1_x}, \dots, v_{(p-1)_x} \in T_x N$. There is another related f -derivation d_X defined by

$$d_X = i_X \circ d + d \circ i_X, \quad (2.3)$$

where d stands for the operator of exterior differentiation. This is of type d_* , i.e., $d_X \circ d_{(M)} = d_{(N)} \circ d_X$.

Note that when $X \in \mathfrak{X}(M)$, then the id_M -derivations i_X and d_X are nothing but the inner product or contraction i_X and the Lie derivative \mathcal{L}_X , respectively.

Let us observe that d_X is an f^* -derivation associated to X in the sense of Pidello and Tulczyjew [35].

If X is a vector field along $f : N \rightarrow M$, from (2.2) and (2.3) we deduce that the map

$$\begin{aligned} d_X &: C^\infty(M) \rightarrow C^\infty(N) \\ F &\rightarrow d_X F \end{aligned}$$

is given by

$$(d_X F)(x) = (i_X dF)(x) = dF(f(x))(X(x)) = X(x)(F), \quad (2.4)$$

for all $x \in N$.

Finally, if $\pi : B \rightarrow M$ is a differentiable fibre bundle, we associate to each π -semi-basic p -form α on B a p -form α^V along π , as follows:

$$\alpha^V(b)(v_{1\pi(b)}, \dots, v_{p\pi(b)}) = \alpha(b)(w_{1b}, \dots, w_{pb}) \quad (2.5)$$

where $v_{i\pi(b)} \in T_{\pi(b)}M$, $i = 1, \dots, p$, and $w_{ib} \in T_bB$ are such that $\pi_*(b)w_{ib} = v_{i\pi(b)}$.

This type of form α^V will be used in Section 5.1.

3. THE GEOMETRY OF JET BUNDLES

In this paper, we work with the first and second order jet bundles $J^1\pi$, $J^2\pi$ of a fiber bundle $\pi : E \rightarrow \mathbb{R}^k$, where E is an $(n+k)$ -dimensional manifold.

If (x^α) are local coordinates on \mathbb{R}^k and (x^α, q^i) are local fiber coordinates on E , we consider standard jet coordinates $(x^\alpha, q^i, v_\alpha^i)$ on $J^1\pi$ and $(x^\alpha, q^i, v_\alpha^i, v_{\alpha\beta}^i)$ on $J^2\pi$, where $1 \leq i \leq n$, $1 \leq \alpha\beta \leq k$, given by

$$\begin{aligned} x^\alpha(j_x^1\phi) &= x^\alpha(x) = x^\alpha, & q^i(j_x^1\phi) &= q^i(\phi(x)), \\ v_\alpha^i(j_x^1\phi) &= \frac{\partial \phi^i}{\partial x^\alpha} \Big|_x, & v_{\alpha\beta}^i(j_x^2\phi) &= \frac{\partial^2 \phi^i}{\partial x^\alpha \partial x^\beta} \Big|_x. \end{aligned}$$

Here, ϕ is a section for π and $j_x^1\phi$, $j_x^2\phi$ are the 1-jet and 2-jet on x , respectively.

For the canonical projections, we use the usual notations

$$\begin{array}{ccccccc} J^2\pi & \xrightarrow{\pi_{2,1}} & J^1\pi & \xrightarrow{\pi_{1,0}} & E & \xrightarrow{\pi_1} & \mathbb{R}^k \\ j_x^2\phi & \rightarrow & j_x^1\phi & \rightarrow & \phi(x) & \rightarrow & x. \end{array}$$

3.1. Canonical vector fields along the projections $\pi_{1,0}$ and $\pi_{2,1}$. The vector field $T_\alpha^{(0)}$ on E along $\pi_{1,0}$ and the vector field $T_\alpha^{(1)}$ on $J^1\pi$ along $\pi_{2,1}$

$$\begin{array}{ccccc} & & T(J^1\pi) & & TE \\ & \nearrow T_\alpha^{(1)} & \downarrow \tau_{J^1\pi} & \nearrow T_\alpha^{(0)} & \downarrow \tau_E \\ J^2\pi & \xrightarrow{\pi_{2,1}} & J^1\pi & \xrightarrow{\pi_{1,0}} & E \end{array}$$

are respectively defined by

$$\begin{aligned} T_\alpha^{(0)}(j_x^1 \phi) &= \phi_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) \in T_{\phi(x)} E \\ T_\alpha^{(1)}(j_x^2 \phi) &= (j^1 \phi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) \in T_{j_x^1 \phi}(J^1 \pi), \end{aligned} \quad (3.1)$$

as we can see from the above diagram.

Their local expressions are given by

$$\begin{aligned} T_\alpha^{(0)} &= \frac{\partial}{\partial x^\alpha} \circ \pi_{1,0} + v_\alpha^i \frac{\partial}{\partial q^i} \circ \pi_{1,0}, \\ T_\alpha^{(1)} &= \frac{\partial}{\partial x^\alpha} \circ \pi_{2,1} + v_\alpha^i \frac{\partial}{\partial q^i} \circ \pi_{2,1} + v_{\alpha,\beta}^i \frac{\partial}{\partial v_\beta^i} \circ \pi_{2,1}. \end{aligned} \quad (3.2)$$

Using (2.4) we have the maps $d_{T_\alpha^{(0)}}$ and $d_{T_\alpha^{(1)}}$

$$\begin{aligned} d_{T_\alpha^{(0)}} &: C^\infty(E) \longrightarrow C^\infty(J^1 \pi) \\ d_{T_\alpha^{(1)}} &: C^\infty(J^1 \pi) \longrightarrow C^\infty(J^2 \pi) \end{aligned}$$

defined by $T_\alpha^{(0)}$, $T_\alpha^{(1)}$, respectively.

From (2.4) and (3.2) one obtains

$$d_{T_\alpha^{(0)}} F = \frac{\partial F}{\partial x^\alpha} \circ \pi_{1,0} + v_\alpha^i \frac{\partial F}{\partial q^i} \circ \pi_{1,0}, \quad (3.3)$$

and

$$d_{T_\alpha^{(1)}} G = \frac{\partial G}{\partial x^\alpha} \circ \pi_{2,1} + v_\alpha^i \frac{\partial G}{\partial q^i} \circ \pi_{2,1} + v_{\alpha,\beta}^i \frac{\partial G}{\partial v_\beta^i} \circ \pi_{2,1}. \quad (3.4)$$

where $F \in C^\infty(E)$ and $G \in C^\infty(J^1 \pi)$.

Remark 3.1. Let us observe that the vector fields $T_\alpha^{(0)}$ and $T_\alpha^{(1)}$ are $(\pi_{2,1}, \pi_{1,0})$ -related in the following sense $(\pi_{1,0})_* \circ T_\alpha^{(1)} = T_\alpha^{(0)} \circ \pi_{2,1}$. \diamond

3.2. Prolongations of vector fields. We recall the prolongations of vector fields from E to $J^1 \pi$ and the prolongation of a vector field along $\pi_{1,0}$, see Saunders [41] (Sections 4.4 and Section 6.4).

Let X be a vector field on E locally given by

$$X = X_\alpha(q, v) \frac{\partial}{\partial x^\alpha} + X^i(q, v) \frac{\partial}{\partial q^i},$$

then its prolongation X^1 is the vector field on $J^1 \pi$ whose local expression is

$$X^1 = X_\alpha \frac{\partial}{\partial x^\alpha} + X^i \frac{\partial}{\partial q^i} + \left(\frac{dX^i}{dx^\alpha} - v_\beta^i \frac{dX_\beta}{dx^\alpha} \right) \frac{\partial}{\partial v_\alpha^i}, \quad (3.5)$$

where d/dx^α denotes the total derivative, that is,

$$\frac{d}{dx^\alpha} = \frac{\partial}{\partial x^\alpha} + v_\alpha^j \frac{\partial}{\partial q^j} \quad 1 \leq \alpha \leq k.$$

Let us observe that $\frac{dF}{dx^\alpha} = T_\alpha^{(0)}(F)$ where $F \in C^\infty(J^1 \pi)$.

Let X be a vector field along $\pi_{1,0}$ and $X^{(1)}$ its first prolongation along $\pi_{2,1}$, which means

$$\begin{array}{ccccc} & & T(J^1 \pi) & & TE \\ & \nearrow X^{(1)} & \downarrow \tau_{J^1 \pi} & \nearrow X & \downarrow \tau_E \\ J^2 \pi & \xrightarrow{\pi_{2,1}} & J^1 \pi & \xrightarrow{\pi_{1,0}} & E \end{array},$$

see Saunders [41], Section 6.4. If X has the local expression

$$X = X_\alpha(x, q, v) \frac{\partial}{\partial x^\alpha} \circ \pi_{1,0} + X^i(x, q, v) \frac{\partial}{\partial q^i} \circ \pi_{1,0}$$

then $X^{(1)}$ is locally given by

$$\begin{aligned} X^{(1)} &= X_\alpha \circ \pi_{2,1} \frac{\partial}{\partial x^\alpha} \circ \pi_{2,1} + X^i \circ \pi_{2,1} \frac{\partial}{\partial q^i} \circ \pi_{2,1} \\ &\quad + \left(d_{T_\alpha^{(1)}}(X^i \circ \pi_{2,1}) - d_{T_\alpha^{(1)}}(X_\beta \circ \pi_{2,1}) v_\beta^i \right) \frac{\partial}{\partial v_\alpha^i} \circ \pi_{2,1} . \end{aligned}$$

If X is a vector field along $\pi_{1,0}$ and π -vertical then it is locally given by

$$X^{(1)} = \left(X^i \frac{\partial}{\partial q^i} \right) \circ \pi_{2,1} + \left(\frac{\partial X^i}{\partial x^\alpha} \circ \pi_{2,1} + v_\alpha^j \frac{\partial X^i}{\partial q^j} \circ \pi_{2,1} + v_{\alpha\beta}^j \frac{\partial X^i}{\partial v_\beta^j} \circ \pi_{2,1} \right) \frac{\partial}{\partial v_\alpha^i} \circ \pi_{2,1} \quad (3.6)$$

According to (2.4), we know that the vector field $X^{(1)}$ along $\pi_{2,1}$ defines a map

$$\begin{aligned} d_{X^{(1)}} : C^\infty(J^1\pi) &\rightarrow C^\infty(J^2\pi) \\ F &\rightarrow d_{X^{(1)}} F . \end{aligned}$$

with local expression

$$d_{X^{(1)}} F = X_\alpha \circ \pi_{2,1} \frac{\partial F}{\partial x^\alpha} \circ \pi_{2,1} + X^i \circ \pi_{2,1} \frac{\partial F}{\partial q^i} \circ \pi_{2,1} + \left(d_{T_\alpha^{(1)}} X^i - v_\beta^i d_{T_\alpha^{(1)}} X_\beta \right) \frac{\partial F}{\partial v_\alpha^i} \circ \pi_{2,1} . \quad (3.7)$$

Remark 3.2. Let X be a π -vertical vector field on E . The vector field $X \circ \pi_{1,0}$ along $\pi_{1,0}$

$$\begin{array}{ccc} & & TE \\ & \nearrow^{X \circ \pi_{1,0}} & \downarrow \\ J^1\pi & \xrightarrow{\pi_{1,0}} & E \end{array}$$

satisfies that

$$(X \circ \pi_{1,0})^{(1)} = X^1 \circ \pi_{2,1} . \quad (3.8)$$

◇

3.3. Vertical endomorphisms. Each 1-form dx^α , $1 \leq \alpha \leq k$, defines a canonical tensor field S_{dx^α} on $J^1\pi$ of type $(1, 1)$, see Saunders [41], page 156, with local expression

$$S_{dx^\alpha} \equiv (dq^i - v_\beta^i dx^\beta) \otimes \frac{\partial}{\partial v_\alpha^i} . \quad (3.9)$$

Throught the paper, we denote S_{dx^α} by S^α .

The vector valued k -form S on $J^1\pi$, defined in [40, 41], whose values are vertical vectors over E , is given in coordinates by

$$S = ((dq^i - v_\beta^i dx^\beta) \wedge d^{k-1}x_\alpha) \otimes \frac{\partial}{\partial v_\alpha^i} \quad (3.10)$$

where

$$d^{k-1}x_\alpha = i_{\partial/\partial x^\alpha} d^k x = (-1)^{\alpha-1} dx^1 \wedge \cdots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge \cdots \wedge dx^k$$

and $d^k x = dx^1 \wedge \cdots \wedge dx^k$ is the standard volume form on \mathbb{R}^k .

From (3.9) and (3.10) we deduce that S and $\{S_{dx^1}, \dots, S_{dx^k}\}$ are related by the formula

$$S = S^\alpha \wedge d^{k-1}x_\alpha .$$

We also have $dx^\alpha \wedge S = -S^\alpha \wedge d^k x$.

3.4. Contact structures and second order partial differential equations.

Let us consider the **Cartan distribution**, which is the $(k + nk)$ -dimensional distribution, given by

$$C(J^1\pi) = \text{Ker } S^1 = \dots = \text{Ker } S^k.$$

From (3.9) we deduce that $X \in C(J^1\pi)$ if, and only if,

$$(dq^i - v_\alpha^i dx^\alpha)(X) = 0,$$

and thus X is locally given by

$$X = X_\alpha \left(\frac{\partial}{\partial x^\alpha} + v_\alpha^i \frac{\partial}{\partial q^i} \right) + X_\alpha^i \frac{\partial}{\partial v_\alpha^i}.$$

Therefore, a local basis for $C(J^1\pi)$ is given by the local $k + nk$ vector fields

$$\frac{\partial}{\partial x^\alpha} + v_\alpha^i \frac{\partial}{\partial q^i}, \quad \frac{\partial}{\partial v_\alpha^i}.$$

We also consider the **contact codistribution**, which is the n -dimensional distribution that represents the annihilator of the Cartan distribution and is given by

$$\Lambda_c^1(J^1\pi) = \{\theta \in \Lambda^1(J^1\pi), (j^1\phi)^*\theta = 0, \forall \phi \in \text{Sec}(\pi)\}.$$

A local basis for $\Lambda_c^1(J^1\pi)$ is the set of canonical 1-forms

$$\delta q^i = dq^i - v_\beta^i dx^\beta, \quad i = 1, \dots, n.$$

Definition 3.3. A k -vector field $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ on $J^1\pi$ is said to be a second order partial differential equation (SOPDE for short) if

$$dx^\alpha(\Gamma_\beta) = \delta_\beta^\alpha, \quad S^\alpha(\Gamma_\beta) = 0,$$

or equivalently,

$$dx^\alpha(\Gamma_\beta) = \delta_\beta^\alpha, \quad \delta q^i(\Gamma_\beta) = 0$$

for all $i = 1 \dots n$, $\alpha, \beta = 1 \dots k$.

Every vector field Γ_α of a SOPDE belongs to the Cartan distribution.

From Definition 3.3 and the expression (3.9) of S^α , we obtain that the local expression of a SOPDE $(\Gamma_1, \dots, \Gamma_k)$ is

$$\Gamma_\alpha(x^\beta, q^i, v_\beta^i) = \frac{\partial}{\partial x^\alpha} + v_\alpha^i \frac{\partial}{\partial q^i} + \Gamma_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i}, \quad 1 \leq \alpha \leq k \quad (3.11)$$

where $\Gamma_{\alpha\beta}^i$ are functions on $J^1\pi$. As a direct consequence of the above local expressions, we deduce that the family of vector fields $\{\Gamma_1, \dots, \Gamma_k\}$ are linearly independent.

Definition 3.4. Let $\phi : \mathbb{R}^k \rightarrow E$ be a section of π , locally given by $\phi(x^\alpha) = (x^\alpha, \phi^i(x^\alpha))$, then the first prolongation $j^1\phi$ of ϕ is the map

$$\begin{aligned} j^1\phi : \mathbb{R}^k &\longrightarrow J^1\pi \\ x &\longrightarrow j_x^1\phi \equiv \left(x^1, \dots, x^k, \phi^i(x^1, \dots, x^k), \frac{\partial \phi^i}{\partial x^\alpha}(x^1, \dots, x^k) \right) \end{aligned} \quad (3.12)$$

for all $\alpha = 1, \dots, k$ and for all $x \in \text{Domain } \phi$.

We will see that the integral sections of a SOPDE are prolongations of sections.

The following proposition has been also proved in Saunders [41],

Proposition 3.5. *A section ψ of π_1 is the 1-jet prolongation of a section of π (in other words it is a **holonomic field**) if, and only, if $\psi^*\theta = 0$ for all $\theta \in \Lambda_c^1(J^1\pi)$.*

Proof. Consider $\psi : \mathbb{R}^k \rightarrow J^1\pi$, $\psi(x^\alpha) = (x^\alpha, \psi^i(x^\alpha), \psi_\beta^i(x^\alpha))$ a section of π_1 and $\theta = \theta_i(dq^i - v_\beta^i dx^\beta) \in \Lambda_c^1(J^1\pi)$. It follows that

$$\psi^*\theta = \theta_i \circ \psi \left(\frac{\partial \psi^i}{\partial x^\beta} - \psi_\beta^i \right) dx^\alpha.$$

Therefore, $\psi^*\theta = 0$ for all $\theta \in \Lambda_c^1(J^1\pi)$ if and only if $\psi_\beta^i = \partial \psi^i / \partial x^\beta$. \square

Now we characterize the integral sections of a SOPDE.

Proposition 3.6. *Let $\mathbf{\Gamma}$ be an integrable k -vector field. Then the following three properties are equivalent:*

- i) $\mathbf{\Gamma}$ is a SOPDE
- ii) The integral sections of $\mathbf{\Gamma}$ are 1-jets prolongations of sections of π .
- iii) There exists a section γ for $\pi_{2,1}$ such that $\Gamma_\alpha = T_\alpha^{(1)} \circ \gamma$.

Proof. i) \Rightarrow ii) Let $\mathbf{\Gamma}$ be an integrable k -vector field and $\psi : \mathbb{R}^k \rightarrow J^1\pi$ an integral section for $\mathbf{\Gamma}$. Then from (2.1) and the local expression (3.11) of Γ_α , we obtain

$$\begin{aligned} (\pi_1 \circ \psi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) &= (\pi_1)_*(\psi(x)) \left(\psi_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) \right) = (\pi_1)_*(\psi(x)) \Gamma_\alpha(\psi(x)) \\ &= \frac{\partial}{\partial x^\alpha} \Big|_{\pi_1(\psi(x))} \end{aligned}$$

which means

$$\frac{\partial}{\partial x^\alpha} \Big|_x (x^\beta \circ \pi_1 \circ \psi) = \delta_\beta^\alpha$$

so $\pi_1 \circ \psi = id_{\mathbb{R}^k}$, i.e., ψ is a local section for π_1 .

We must prove that $\psi = j^1\phi$ where ϕ is a section of π . To this end we use Proposition 3.5, showing that $\psi^*\theta = 0$ for all $\theta \in \Lambda_C^1(J^1\pi)$.

Let us assume that ψ is an integral section passing through $j_x^1\varphi$, that is $\psi(x) = j_x^1\varphi$.

Now, since $\mathbf{\Gamma}$ is a SOPDE and $\theta \in \Lambda_C^1(J^1\pi)$, then $i_{\Gamma_\alpha}\theta = 0$. Thus,

$$\begin{aligned} 0 &= i_{\Gamma_\alpha}\theta(j_x^1\varphi) = \theta(j_x^1\varphi)\Gamma_\alpha(j_x^1\varphi) = \theta(j_x^1\varphi)\Gamma_\alpha(\psi(x)) \\ &= \theta(\psi(x)) \left(\psi_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) \right) = (\psi^*\theta)(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) \end{aligned}$$

So, we have proved $\psi^*\theta = 0$ for all $\theta \in \Lambda_C^1(J^1\pi)$.

ii) \Rightarrow iii) We define

$$\begin{aligned} \gamma : J^1\pi &\rightarrow J^2\pi \\ j_x^1\sigma &\rightarrow \gamma(j_x^1\sigma) = j_x^2\varphi, \end{aligned}$$

where $j^1\varphi$ is an integral section of $\mathbf{\Gamma}$ passing through $j_x^1\sigma$ (i.e., $j_x^1\varphi = j^1\varphi(x) = j_x^1\sigma$).

Then

$$(\pi_{2,1} \circ \gamma)(j_x^1\sigma) = \pi_{2,1}(\gamma(j_x^1\sigma)) = \pi_{2,1}(j_x^2\varphi) = j_x^1\varphi = j_x^1\sigma.$$

It follows that the map γ so defined is a section for $\pi_{2,1}$. Moreover, the vector field Γ_α can be expressed as $\Gamma_\alpha = T_\alpha^{(1)} \circ \gamma$, in fact,

$$\Gamma_\alpha(j_x^1\sigma) = (j^1\varphi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) = T_\alpha^{(1)}(j_x^2\varphi) = (T_\alpha^{(1)} \circ \gamma)(j_x^1\sigma).$$

iii) \Rightarrow i) Let γ be a section for $\pi_{2,1}$. We must prove that γ defines a SOPDE by composition with $T_\alpha^{(1)}$. Since

$$\tau_{J^1\pi} \circ \Gamma_\alpha = \tau_{J^1\pi} \circ T_\alpha^{(1)} \circ \gamma = \pi_{2,1} \circ \gamma = id_{J^1\pi},$$

where $\tau_{J^1\pi}: T(J^1\pi) \rightarrow J^1\pi$ is the canonical projection, then $\Gamma_\alpha = T_\alpha^{(1)} \circ \gamma$ is a vector field on $J^1\pi$. Moreover, $\mathbf{\Gamma}$ is a SOPDE

$$dx^\beta(\Gamma_\alpha)(j_x^1\sigma) = dx^\beta(T_\alpha^{(1)} \circ \gamma)(j_x^1\sigma) = \delta_\beta^\alpha$$

and

$$\begin{aligned} \delta q^i(\Gamma_\alpha)(j_x^1\sigma) &= \delta q^i(j_x^1\sigma) \left((T_\alpha^{(1)} \circ \gamma)(j_x^1\sigma) \right) = \delta q^i(j_x^1\sigma) \left(T_\alpha^{(1)}(j_x^2\varphi) \right) \\ &= v_\alpha^i(j_x^2\varphi) - v_\alpha^i(j_x^1\sigma) = 0, \end{aligned}$$

where the last identity is true because γ is a section for $\pi_{2,1}$ so $j_x^1\sigma = (\pi_{2,1} \circ \gamma)(j_x^1\sigma) = \pi_{2,1}(j_x^2\varphi) = j_x^1\varphi$. \square

If $j^1\phi$ is an integral section of a SOPDE $\mathbf{\Gamma}$, then ϕ is called a *solution of $\mathbf{\Gamma}$* .

From (2.1) and (3.11), we deduce that ϕ is a solution of $\mathbf{\Gamma}$ if, and only if, $q^i \circ \phi = \phi^i$ is a solution to the following system of second order partial differential equations

$$\frac{\partial^2 \phi^i}{\partial x^\alpha \partial x^\beta} \Big|_x = \Gamma_{\alpha\beta}^i \left(x, \phi^i(x), \frac{\partial \phi^i}{\partial x^\gamma} \right). \quad (3.13)$$

where $1 \leq i \leq n$ and $1 \leq \alpha, \beta \leq k$.

The integrability conditions for the system (3.13) requires that the k -dimensional distribution induced by the SOPDE $\mathbf{\Gamma}$, $H_0 = \text{span}\{\Gamma_1, \dots, \Gamma_k\}$ is integrable. The local expression (3.11) for a SOPDE $\mathbf{\Gamma}$ shows that $[\Gamma_\alpha, \Gamma_\beta] = A_{\alpha\beta}^\gamma \Gamma_\gamma$ if, and only if, $A_{\alpha\beta}^\gamma = 0$. Therefore, for a SOPDE $\mathbf{\Gamma}$, we assume throughout the paper the following integrability conditions

$$\Gamma_{\alpha\beta}^i = \Gamma_{\beta\alpha}^i, \quad \Gamma_\alpha(\Gamma_{\beta\gamma}^i) = \Gamma_\beta(\Gamma_{\alpha\gamma}^i), \quad (3.14)$$

which are equivalent to the condition $[\Gamma_\alpha, \Gamma_\beta] = 0$ for all $\alpha, \beta = 1, \dots, k$.

4. LAGRANGIAN FIELD THEORY

4.1. Poincarè-Cartan 1-forms. Let $L: J^1\pi \rightarrow \mathbb{R}$ be a Lagrangian.

For each $\alpha = 1 \dots k$ we define the *Poincarè-Cartan 1-forms* Θ_L^α on $J^1\pi$ as the 1-forms

$$\Theta_L^\alpha = L dx^\alpha + dL \circ S^\alpha, \quad 1 \leq \alpha \leq k.$$

Their local expressions are given by

$$\Theta_L^\alpha = \left(L \delta_\beta^\alpha - \frac{\partial L}{\partial v_\alpha^i} v_\beta^i \right) dx^\beta + \frac{\partial L}{\partial v_\alpha^i} dq^i, \quad (4.1)$$

or equivalently

$$\Theta_L^\alpha = \frac{\partial L}{\partial v_\alpha^i} (dq^i - v_\beta^i dx^\beta) + L dx^\alpha = \frac{\partial L}{\partial v_\alpha^i} \delta q^i + L dx^\alpha. \quad (4.2)$$

Let us observe that the fact of working with a fiber bundle over \mathbb{R}^k allows us to introduce the tensors S^α and consequently the 1-forms Θ_L^α .

It is known that a Lagrangian L induces a k -form Θ_L , called the Cartan form in first-order field theories, Saunders [41, 40], which is given by

$$\Theta_L = L d^k x + dL \circ S.$$

with local expression

$$\Theta_L = \frac{\partial L}{\partial v_\alpha^i} (dq^i - v_\beta^i dx^\beta) \wedge \left(i_{\frac{\partial}{\partial x^\alpha}} \wedge d^k x \right) + L d^k x .$$

The relationship between the Cartan form and the Poincaré-Cartan 1-forms is given by the following identity

$$\Theta_L = \Theta_L^\alpha \wedge d^{k-1} x_\alpha + (1 - k) L d^k x .$$

4.2. Euler-Lagrange field equations. Let $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ be an integrable SOPDE. A direct computation, using (3.11), (3.14) and (4.2), gives the formula

$$\mathcal{L}_{\Gamma_\alpha} \Theta_L^\alpha = dL + \left(\Gamma_\alpha \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} \right) (dq^i - v_\beta^i dx^\beta), \quad (4.3)$$

and proves the following lemma.

Lemma 4.1. *Let $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ be an integrable SOPDE satisfying*

$$\Gamma_\alpha \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n .$$

If $j^1 \phi$ is an integral section of Γ , then ϕ is a solution to the Euler-Lagrange equations

$$\frac{\partial^2 L}{\partial x^\alpha \partial v_\alpha^i} \Big|_{j_x^1 \phi} + \frac{\partial \phi^j}{\partial x^\alpha} \Big|_x \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} \Big|_{j_x^1 \phi} + \frac{\partial^2 \phi^j}{\partial x^\alpha \partial x^\beta} \Big|_x \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i} \Big|_{j_x^1 \phi} = \frac{\partial L}{\partial q^i} \Big|_{j_x^1 \phi}, \quad (4.4)$$

where $x \in \text{Domain } \phi$. Equations (4.4) are usually written as follows

$$\frac{\partial}{\partial x^\alpha} \Big|_x \left(\frac{\partial L}{\partial v_\alpha^i} \circ j^1 \phi \right) = \frac{\partial L}{\partial q^i} \Big|_{j_x^1 \phi}, \quad 1 \leq i \leq k .$$

From (4.3) we deduce the following proposition:

Proposition 4.2. *Let Γ be an integrable SOPDE, then*

$$\mathcal{L}_{\Gamma_\alpha} \Theta_L^\alpha = dL \quad (4.5)$$

if and only if

$$\Gamma_\alpha \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n .$$

Thus, we have the following result

Corollary 4.3. *Let Γ be an integrable SOPDE; if $j^1 \phi$ is solution to (4.5), that is $\mathcal{L}_{\Gamma_\alpha} \Theta_L^\alpha(j^1 \phi) = dL(j^1 \phi)$, then ϕ is solution to the Euler-Lagrange equations.*

From (4.5) and the identity $i_{\Gamma_\alpha} \Theta_L^\alpha = kL$ we deduce the following proposition:

Proposition 4.4. *Let Γ be a SOPDE, then the equation (4.5) is equivalent to the equation*

$$i_{\Gamma_\alpha} \Omega_L^\alpha = (k - 1) dL ,$$

where $\Omega_L^\alpha = -d\Theta_L^\alpha$ will be called the Poincaré-Cartan 2-forms.

□

Taking into account the above results we are able prove a theorem that gives us a new Lagrangian field formulation for a bundle $E \rightarrow \mathbb{R}^k$.

First we recall that a Lagrangian $L : J^1\pi \rightarrow \mathbb{R}$ is said to be regular if the matrix

$$\left(\frac{\partial^2 L}{\partial v_\alpha^j \partial v_\beta^i} \right)_{1 \leq \alpha, \beta \leq k, 1 \leq j, i \leq n}$$

is not singular, where $1 \leq \alpha, \beta \leq k, 1 \leq j, i \leq n$.

Theorem 4.5. *Let $\mathbf{X} = (X_1, \dots, X_k)$ be a k -vector on $J^1\pi$ such that*

$$dx^\alpha(X_\beta) = \delta_\beta^\alpha, \quad i_{X_\alpha} \Omega_L^\alpha = (k-1)dL, \quad (4.6)$$

- (i) *If L is regular then $\mathbf{X} = (X_1, \dots, X_k)$ is a SOPDE. If \mathbf{X} is integrable and $\phi : U_0 \subset \mathbb{R}^k \rightarrow E$ is a solution of \mathbf{X} then ϕ is a solution to the Euler-Lagrange equations (4.4).*
- (ii) *Now, if $\mathbf{X} = (X_1, \dots, X_k)$ is integrable and $j^1\phi$ is an integral section of \mathbf{X} , then ϕ is a solution to the Euler-Lagrange equations (4.4).*

Proof. (i) If we write each X_α in local coordinates as

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + X_\alpha^i \frac{\partial}{\partial q^i} + X_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i}, \quad 1 \leq \alpha \leq k,$$

then, from (4.1) and (4.6), we obtain

$$\begin{aligned} (1) \quad X_\alpha \left(L\delta_\beta^\alpha - \frac{\partial L}{\partial v_\alpha^i} v_\beta^i \right) + (v_\alpha^i - X_\alpha^i) \frac{\partial^2 L}{\partial x^\beta \partial v_\alpha^i} &= \frac{\partial L}{\partial x^\beta} \\ (2) \quad X_\alpha \left(\frac{\partial L}{\partial v_\alpha^j} \right) + (v_\alpha^i - X_\alpha^i) \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} &= \frac{\partial L}{\partial q^j} \\ (3) \quad (v_\alpha^i - X_\alpha^i) \frac{\partial^2 L}{\partial v_\alpha^j \partial v_\beta^i} &= 0. \end{aligned}$$

Using that L is regular we deduce from (3) that $X_\alpha^i = v_\alpha^i$.

Since \mathbf{X} is a SOPDE and we assume it to be integrable, its integral sections are prolongations $j^1\phi$ of sections ϕ of π . Then, from

$$X_\alpha(j^1\phi_x) = (j^1\phi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right)$$

we deduce

$$X_\alpha^i \circ j^1\phi = v_\alpha^i \circ j^1\phi = \frac{\partial \phi^i}{\partial x^\alpha}, \quad X_{\alpha\beta}^i \circ j^1\phi = \frac{\partial^2 \phi^i}{\partial x^\alpha \partial x^\beta}, \quad (4.7)$$

and from (2), we obtain

$$\frac{\partial^2 L}{\partial x^\alpha \partial v_\alpha^i} + \frac{\partial \phi^j}{\partial x^\alpha} \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i} + \frac{\partial^2 \phi^j}{\partial x^\alpha \partial x^\beta} \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i} = \frac{\partial L}{\partial q^i}.$$

which means that ϕ is solution to the Euler-Lagrange equations (4.4).

(ii) If $j^1\phi$ is an integral section of \mathbf{X} , then from the local expression (3.12) of $j^1\phi$ we obtain the equations (4.7). Thus, from equation (2) and (4.7) we deduce that ϕ is a solution to the Euler-Lagrange equations (4.4). □

Remark 4.6. For $k = 1$, that is the fibration is $E \rightarrow \mathbb{R}$, we obtain the well-known dynamical equation $i_{\Gamma}d\Theta = 0$, where Γ is a second order differential equation SODE, see [3].

In the case $E = \mathbb{R} \times Q \rightarrow \mathbb{R}$, equations (4.6) can be written as the dynamical equations

$$dt(X) = 1, \quad i_X \Omega_L = 0$$

where Ω_L is the Poincare-Cartan 2-form defined by the Lagrangian $L: \mathbb{R} \times TQ \rightarrow \mathbb{R}$. These equations coincide with the cosymplectic formulation of the non-autonomous mechanics, see [8, 27].

The trivial case $E = \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$ correspond to the k -cosymplectic formalism described in [26], which will be analyzed in next section. \diamond

Example 4.7. *The Klein-Gordon equation*

The equation of a scalar field ϕ (for instance the gravitational field) which acts on the four-dimensional space-time is [20]

$$(\square + m^2)\phi = F'(\phi) \quad (4.8)$$

where m is the mass of the particle over which the field acts, F is a scalar function such that $F(\phi) - \frac{1}{2}m^2\phi^2$ is the potential energy of the particle of mass m , and \square is the Laplace-Beltrami operator given by

$$\square\phi := \operatorname{div} \operatorname{grad}\phi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{\alpha\beta} \frac{\partial \phi}{\partial x^\beta}),$$

$(g_{\alpha\beta})$ being a pseudo-riemannian metric tensor in the four-dimensional space-time of signature $(-+++)$.

Now we consider the trivial bundle $\pi: E = \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$, with coordinates (x^1, \dots, x^4, q) on E , and $(x^1, \dots, x^4, q, v_1, \dots, v_4)$ the induced coordinates on $J^1\pi$.

Let L be the Lagrangian $L: J^1\pi \rightarrow \mathbb{R}$ defined by

$$L(x^1, \dots, x^4, q, v_1, \dots, v_4) = \sqrt{-g}(F(q) - \frac{1}{2}m^2q^2 + \frac{1}{2}g^{\alpha\beta}v_\alpha v_\beta),$$

which is regular.

Let us assume that $\mathbf{X} = (X_1, X_2, X_3, X_4)$ is an integrable 4-vector field on $J^1\pi$ solution to the equations (4.6), that is

$$dx^\alpha(X_\beta) = \delta_\beta^\alpha, \quad i_{X_1}\Omega_L^1 + i_{X_2}\Omega_L^2 + i_{X_3}\Omega_L^3 + i_{X_4}\Omega_L^4 = 3dL \quad (4.9)$$

From (4.9) we deduce

$$X_\alpha \left(\frac{\partial L}{\partial v_\alpha} \right) = \frac{\partial L}{\partial q},$$

which is equivalent to

$$X_\alpha(\sqrt{-g}g^{\alpha\beta}v_\beta) = \frac{\partial L}{\partial q} = \sqrt{-g}(F'(q) - m^2q).$$

Since L is regular, then \mathbf{X} is a SOPDE, and if $j^1\psi$ is an integral section of \mathbf{X} , then

$$(j^1\psi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) (\sqrt{-g}g^{\alpha\beta}v_\beta) = \sqrt{-g}(F'(\psi(x)) - m^2\psi(x))$$

which can be written as

$$0 = \sqrt{-g} \frac{\partial}{\partial x^\alpha} \left(g^{\alpha\beta} \frac{\partial \psi}{\partial x^\beta} \right) - \sqrt{-g}(F'(\psi) - m^2\psi)$$

and thus we obtain that ψ is a solution to the scalar field equation (4.8).

Remark 4.8. Some particular cases of the scalar field equation (4.8) are:

- (i) If $F = 0$, we obtain the linear scalar field equation.
- (ii) If $F(q) = m^2 q^2$, we obtain the Klein-Gordon equation, see [18],

$$(\square + m^2)\phi = 0.$$

◇

4.3. A particular case: the k -cosymplectic formalism. In this section we show that the k -cosymplectic Lagrangian formulation, introduced in [26], is a particular case of the Lagrangian formulation proposed in Section 4.2.

Throughout this section we consider the trivial bundle $E = \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$. In this case, the manifold $J^1\pi$ of 1-jets of sections of the trivial bundle $\pi : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^k \times T_k^1 Q$, where $T_k^1 Q$ denotes the Whitney sum $TQ \oplus \dots \oplus TQ$ of k copies of TQ .

The diffeomorphism is given by

$$\begin{aligned} J^1\pi &\rightarrow \mathbb{R}^k \times T_k^1 Q \\ j_x^1 \phi &= j_x^1 (Id_{\mathbb{R}^k}, \phi_Q) \rightarrow (x, v_1, \dots, v_k) \end{aligned}$$

where $\phi_Q : \mathbb{R}^k \xrightarrow{\phi} \mathbb{R}^k \times Q \xrightarrow{\pi_Q} Q$, and

$$v_\alpha = (\phi_Q)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right), \quad 1 \leq \alpha \leq k.$$

Now we recall the k -cosymplectic Lagrangian formalism, beginning with the necessary geometric elements

- The Liouville vector field Δ on $\mathbb{R}^k \times T_k^1 Q$ is the infinitesimal generator of the following flow

$$\begin{aligned} \mathbb{R} \times (\mathbb{R}^k \times T_k^1 Q) &\rightarrow \mathbb{R}^k \times T_k^1 Q \\ (s, (t, v_{1q}, \dots, v_{kq})) &\rightarrow (t, e^s v_{1q}, \dots, e^s v_{kq}) \end{aligned}$$

and in local coordinates it has the form $\Delta = v_\alpha^i \frac{\partial}{\partial v_\alpha^i}$.

- The canonical vector field Δ_β^α , $1 \leq \alpha, \beta \leq k$ is the vector field on $\mathbb{R}^k \times T_k^1 Q$ defined by

$$\Delta_\beta^\alpha(x, v_{1q}, \dots, v_{kq}) = \frac{d}{ds} \Big|_0 \left(x, v_{1q}, \dots, v_{\alpha-1q}, v_{\alpha q} + s v_{\beta q}, v_{\alpha+1q}, \dots, v_{kq} \right)$$

with local expression $\Delta_\beta^\alpha = v_\beta^i \frac{\partial}{\partial v_\alpha^i}$. Let us observe that $\Delta = \Delta_\alpha^\alpha$.

- For a Lagrangian $L : \mathbb{R}^k \times T_k^1 Q \rightarrow \mathbb{R}$, the *energy function* is defined as $E_L = \Delta(L) - L$ and its local expression is

$$E_L = v_\alpha^i \frac{\partial L}{\partial v_\alpha^i} - L. \quad (4.10)$$

- The canonical k -tangent structure on $T_k^1 Q$ is the family $\{J^1, \dots, J^k\}$ of tensor fields locally given by

$$J_\alpha = \frac{\partial}{\partial v_\alpha^i} \otimes dq^i.$$

The *natural extension* J^α of the tensor fields J^α on $T_k^1 Q$ to $\mathbb{R}^k \times T_k^1 Q$ will be denoted by \tilde{J}^α .

- The Poincaré-Cartan 1-forms introduced in [26] are defined as follows

$$\theta_L^\alpha = dL \circ \tilde{J}^\alpha \quad 1 \leq \alpha \leq k,$$

and they have the local expression

$$\theta_L^\alpha = \frac{\partial L}{\partial v_\alpha^i} dq^i. \quad (4.11)$$

The corresponding Poincaré-Cartan 2-forms are $\omega_L^\alpha = -d\theta_L^\alpha$. From (4.1) and (4.11) we deduce that the relationship between the Poincaré-Cartan 1-forms Θ_L^α and θ_L^α is given by the following equation

$$\Theta_L^\alpha = \theta_L^\alpha + (\delta_\beta^\alpha L - \Delta_\beta^\alpha(L)) dx^\beta. \quad (4.12)$$

As a consequence of (4.12), the solutions (X_1, \dots, X_k) of our geometric field equations

$$dx^\alpha(X_\beta) = \delta_\beta^\alpha, \quad i_{X_\alpha} \Omega_L^\alpha = (k-1)dL$$

coincide with the solutions of the k -cosymplectic field equations

$$dx^\alpha(X_\beta) = \delta_\beta^\alpha, \quad 1 \leq \alpha, \beta \leq k,$$

$$i_{X_\alpha} \omega_L^\alpha = dE_L + \frac{\partial L}{\partial x^\alpha} dx^\alpha.$$

introduced in [26], and also the corresponding integral sections, if they exist.

5. SYMMETRIES AND CONSERVATION LAWS

The set of k -vector fields solution to the equation (4.6) will be denoted by $\mathfrak{X}_L^k(J^1\pi)$. As a consequence of Propositions 4.2 and 4.4 we have that an integrable SOPDE $\mathbf{\Gamma}$ belongs to $\mathfrak{X}_L^k(J^1\pi)$ if, and only if, $\mathcal{L}_{\Gamma_\alpha} \Theta_L^\alpha = dL$.

Definition 5.1. A conservation law (or a conserved quantity) for the Euler-Lagrange equations (4.4) is a map $G = (G^1, \dots, G^k): J^1\pi \rightarrow \mathbb{R}^k$ such that the divergence of

$$G \circ j^1\phi = (G^1 \circ j^1\phi, \dots, G^k \circ j^1\phi): U \subset \mathbb{R}^k \rightarrow \mathbb{R}^k$$

is zero for every section $\phi: U \subset \mathbb{R}^k \rightarrow E$, solution to the Euler-Lagrange equations (4.4), which means that for all $x \in U \subset \mathbb{R}^k$ we have

$$0 = [Div(G \circ j^1\phi)](x) = \frac{\partial(G^\alpha \circ j^1\phi)}{\partial x^\alpha} \Big|_x = j^1\phi_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) (G^\alpha) = T_\alpha^{(1)}(j_x^2\phi)(G^\alpha).$$

We can characterize conservation laws for the Euler-Lagrange equations in terms of the SOPDEs in $\mathfrak{X}_L^k(J^1\pi)$.

Proposition 5.2. The map $G = (G^1, \dots, G^k): J^1\pi \rightarrow \mathbb{R}^k$ defines a conservation law for the Euler-Lagrange equations (4.4) if, and only if, for every integrable SOPDE $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_k) \in \mathfrak{X}_L^k(J^1\pi)$ we have that

$$\mathcal{L}_{\Gamma_\alpha} G^\alpha = 0.$$

Proof. Let $j_x^1\phi$ an arbitrary point of $J^1\pi$. Since $\mathbf{\Gamma}$ is an integrable SOPDE, let us denote by $j^1\psi$ the integral section of $\mathbf{\Gamma}$ passing through by $j_x^1\phi$, which means

$$j^1\psi(0) = j_x^1\psi = j_x^1\phi, \quad \Gamma_\alpha(j_x^1\psi) = (j^1\psi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right), \quad x \in \text{Domain } \psi$$

Since $\mathbf{\Gamma} \in \mathfrak{X}_L^k(J^1\pi)$, and ψ is an integral section of $\mathbf{\Gamma}$ then ψ is a solution to the Euler-Lagrange equations (4.4). As $G = (G^1, \dots, G^k)$ is a conservation law, then by hypothesis

$$\frac{\partial(G^\alpha \circ j^1\psi)}{\partial x^\alpha} \Big|_0 = 0,$$

and therefore we deduce

$$\mathcal{L}_{\Gamma_\alpha} G^\alpha(j_x^1 \phi) = \Gamma_\alpha(j_0^1 \psi)(G^\alpha) = (j^1 \psi)_*(0) \left(\frac{\partial}{\partial x^\alpha} \Big|_0 \right) (G^\alpha) = \frac{\partial(G^\alpha \circ j^1 \psi)}{\partial x^\alpha} \Big|_0 = 0.$$

Conversely, we must prove that

$$\frac{\partial(G^\alpha \circ j^1 \phi)}{\partial x^\alpha} \Big|_x = 0,$$

for all sections $\phi : W \subset \mathbb{R}^k \rightarrow E$, which are solutions to the Euler-Lagrange equations (4.4).

Since $j^1 \phi|_W : W \subset \mathbb{R}^k \rightarrow J^1 \pi$ is an injective immersion ($j^1 \phi$ is a section and hence its image is an embedded submanifold), we can define a k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ in $j^1 \phi(W)$ as follows:

$$X_\alpha(j_x^1 \phi) = (j^1 \phi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) = \frac{\partial}{\partial x^\alpha} \Big|_{j_x^1 \phi} + \frac{\partial \phi^i}{\partial x^\alpha} \frac{\partial}{\partial q^i} \Big|_{j_x^1 \phi} + \frac{\partial^2 \phi^i}{\partial x^\alpha \partial x^\beta} \frac{\partial}{\partial v_\beta^i} \Big|_{j_x^1 \phi}$$

and so $j^1 \phi$ is an integral section of \mathbf{X} and \mathbf{X} it is an integrable SOPDE on $j^1 \phi(W)$.

Now we prove that $\mathbf{X} \in \mathfrak{X}_L(j^1 \phi(W))$. A direct computation shows that

$$\left(X_\alpha \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} \right) \Big|_{j^1 \phi(W)} = 0$$

Now, since X is an integrable SOPDE, from Proposition 4.2 and Proposition 4.4 we deduce that X is a solution to the equations (4.6), and then $\mathbf{X} \in \mathfrak{X}_L(j^1 \phi(W))$.

The following identities finish the proof:

$$\frac{\partial(G^\alpha \circ j^1 \phi)}{\partial x^\alpha} \Big|_x = (j^1 \phi)_*(x) \left(\frac{\partial}{\partial x^\alpha} \Big|_x \right) (G^\alpha) = X_\alpha(j_x^1 \phi)(G^\alpha) = \mathcal{L}_{X_\alpha} G^\alpha(j_x^1 \phi) = 0.$$

□

5.1. Generalized symmetries. Noether's Theorem. In this section, we introduce the (generalized) symmetries of the Lagrangian and we prove a Noether's theorem which associates to each symmetry a conservation law.

The following proposition can be seen as a motivation of the condition (5.6) in the definition of generalized symmetry, and it is also a generalization of Proposition 3.15 in [38].

Proposition 5.3. *Let X be a π -vertical vector field on E . If there exist functions $g^\alpha : E \rightarrow \mathbb{R}$, $1 \leq \alpha \leq k$ such that*

$$X^1(L) = d_{T_\alpha^{(0)}} g^\alpha$$

then the functions $G^\alpha = (\pi_{1,0})^ g^\alpha - \Theta_L^\alpha(X^1)$ define a conservation law.*

Proof. Let us observe that locally

$$G^\alpha = g^\alpha \circ \pi_{1,0} - (X^i \circ \pi_{1,0}) \frac{\partial L}{\partial v_\alpha^i}.$$

Then, taking into account (3.3) and (3.5), we deduce that for every solution ϕ of the Euler-Lagrange equations (4.4) we have

$$\frac{\partial(G^\alpha \circ j^1 \phi)}{\partial x^\alpha} \Big|_x = \frac{\partial}{\partial x^\alpha} \Big|_x \left(g^\alpha \circ \phi - (X^i \circ \phi) \left(\frac{\partial L}{\partial v_\alpha^i} \circ j^1 \phi \right) \right) = [d_{T_\alpha^{(0)}} g^\alpha - X^1(L)](j_x^1 \phi) = 0$$

so, (G^1, \dots, G^k) defines a conservation law.

□

The Euler-Lagrange form δL is the 1-form on $J^2\pi$ given by

$$\delta L = d_{T_\alpha^{(1)}} \Theta_L^\alpha - \pi_{2,1}^* dL$$

with local expression

$$\delta L = \left(T_\alpha^{(1)} \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} \circ \pi_{2,1} \right) (dq^i - v_\beta^i dx^\beta).$$

This is a $\pi_{2,0}$ -semi-basic form, and we consider its associated form $(\delta L)^V$ along $\pi_{2,0}$, see (2.5), with local expression

$$\left(T_\alpha^{(1)} \left(\frac{\partial L}{\partial v_\alpha^i} \right) - \frac{\partial L}{\partial q^i} \circ \pi_{2,1} \right) (dq^i \circ \pi_{2,0} - v_\beta^i dx^\beta \circ \pi_{2,0}). \quad (5.1)$$

The 1-forms Θ_L^α are $\pi_{1,0}$ -semi-basic 1-form and its associated forms $(\Theta_L^\alpha)^V$ along $\pi_{1,0}$ are locally given by

$$(\Theta_L^\alpha)^V = (L\delta_\beta^\alpha - \frac{\partial L}{\partial v_\alpha^i} v_\beta^i) dx^\beta \circ \pi_{1,0} + \frac{\partial L}{\partial v_\alpha^i} dq^i \circ \pi_{1,0}. \quad (5.2)$$

The following lemma will be useful in the study of generalized symmetries.

Lemma 5.4. *Let X be a π -vertical vector field X along $\pi_{1,0}$. Then*

- (i) *If there exists functions $G^\alpha : J^1\pi \rightarrow \mathbb{R}$, $\alpha = 1, \dots, k$, such that*

$$d_{T_\alpha^{(1)}} G^\alpha(j^1\phi) = -(\delta L)^V(X \circ \pi_{2,1})(j^1\phi)$$

for any ϕ solution to the Euler-Lagrange equations, then (G^1, \dots, G^k) is a conservation law.

- (ii) *The following identity holds*

$$d_{X^{(1)}} L = -(\delta L)^V(X \circ \pi_{2,1}) + d_{T_\alpha^{(1)}} [(\Theta_L^\alpha)^V(X)]. \quad (5.3)$$

Proof.

- (i) From (3.1) we have

$$d_{T_\alpha^{(1)}} G^\alpha(j_x^2\phi) = \frac{\partial(G^\alpha \circ j^1\phi)}{\partial x^\alpha} \Big|_x \quad (5.4)$$

for any $j_x^2\phi \in J^2\pi$.

Since X is locally given by

$$X = X^i(x, q, v) \frac{\partial}{\partial q^i} \circ \pi_{1,0}$$

then $X \circ \pi_{2,1}$ is locally given by

$$X \circ \pi_{2,1} = \left(X^i(x, q, v) \frac{\partial}{\partial q^i} \circ \pi_{1,0} \right) \circ \pi_{2,1} = X^i(x, q, v) \circ \pi_{2,1} \frac{\partial}{\partial q^i} \circ \pi_{2,0}.$$

From the above local expressions of $X \circ \pi_{2,1}$ and the local expression (5.1) of $(\delta L)^V$ we obtain

$$-(\delta L)^V(X \circ \pi_{2,1})(j_x^2\phi) = - \left(\frac{\partial}{\partial x^\alpha} \Big|_x \left(\frac{\partial L}{\partial v_\alpha^i} \circ j^1\phi \right) - \frac{\partial L}{\partial q^i} \Big|_{j_x^1\phi} \right) X^i(j_x^1\phi) \quad (5.5)$$

Now from (5.4) and (5.5) we obtain that if ϕ is a solution to the Euler-Lagrange equations then

$$\frac{\partial(G^\alpha \circ j^1\phi)}{\partial x^\alpha} \Big|_x = 0.$$

- (ii) A direct computation using (3.6), (3.7), (5.1) and (5.2) proves the identity (5.3).

□

Some classes of symmetries depend only on the variables (coordinates) in E . In this section we consider v_α^i -dependent infinitesimal transformations which can be regarded as vector fields X along $\pi_{1,0}$. The following definition is motivated by Proposition 5.3.

Definition 5.5. A π -vertical vector field X along $\pi_{1,0}$ is called a (generalized) symmetry if there exists a map $(F^1, \dots, F^k): J^1\pi \rightarrow \mathbb{R}^k$ such that

$$d_{X^{(1)}}L(j_x^2\phi) = d_{T_\alpha^{(1)}}F^\alpha(j_x^2\phi) \quad (5.6)$$

for every solution ϕ to the Euler-Lagrange equations.

The following version of Noether's Theorem associates to each symmetry of the Lagrangian, in the sense given above, a conservation law.

Theorem 5.6. Let X be a symmetry of the Lagrangian L then the map

$$G = (G^1, \dots, G^k): J^1\pi \rightarrow \mathbb{R}^k$$

given by

$$G^\alpha = F^\alpha - (\Theta_L^\alpha)^V(X)$$

defines a conservation law.

Proof. Let X be a symmetry of the Lagrangian L . Then from (5.3) we get

$$d_{T_\alpha^{(1)}}[F^\alpha - (\Theta_L^\alpha)^V(X)] = -(\delta L)^V(X \circ \pi_{2,1})$$

and from Lemma 5.4, the functions $G^\alpha = F^\alpha - (\Theta_L^\alpha)^V(X)$ define a conservation law.

□

Example 5.7. Consider the homogeneous isotropic 2-dimensional wave equation

$$\partial_{11}\phi - c^2\partial_{22}\phi - c^2\partial_{33}\phi = 0, \quad (5.7)$$

where $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a solution, and defines a section of the trivial bundle $\pi: E = \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$.

Since $J^1\pi = \mathbb{R}^3 \times T_3^1\mathbb{R}$, equation (5.7) can be described as the Euler-Lagrange equation for the Lagrangian $L: \mathbb{R}^3 \times T_3^1\mathbb{R} \rightarrow \mathbb{R}$ given by

$$L(x, q, v) = \frac{1}{2}((v_1)^2 - c^2(v_2)^2 - c^2(v_3)^2).$$

In this case, for simplicity we consider the case $c = 1$.

With the vector field $X = v_1 \frac{\partial}{\partial q} \circ \pi_{1,0}$ along $\pi_{1,0}$ and the functions on $J^1\pi$

$$F^1(v_1, v_2, v_3) = -c^2(v_2)^2 - c^2(v_3)^2, \quad F^2(v_1, v_2, v_3) = c^2v_1v_2, \quad F^3(v_1, v_2, v_3) = c^2v_1v_3.$$

using Theorem 5.6, we deduce that the following functions

$$\begin{aligned} G^1 &= F^1 - (\Theta_L^1)^V(X) = -c^2(v_2)^2 - c^2(v_3)^2 - (v_1)^2 \\ G^2 &= F^2 - (\Theta_L^2)^V(X) = 2c^2v_1v_2 \\ G^3 &= F^3 - (\Theta_L^3)^V(X) = 2c^2v_1v_3 \end{aligned}$$

define a conservation law.

5.2. Variational symmetries. In this section we consider the trivial bundle $\pi : E = \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$, and we recall some results of variational symmetries of the Euler-Lagrange equations that can be found in Olver's book [34].

Let us remember that the solution of the Euler-Lagrange equation (4.4) can be obtained as the extremals of the functional

$$\mathcal{L}(\phi) = \int_{\Omega_0} (L \circ j^1 \phi)(x) d^k x,$$

where $d^k x = dx^1 \wedge \cdots \wedge dx^k$ is the volume form on \mathbb{R}^k . Roughly speaking, a variational symmetry is a diffeomorphism that leaves the variational integral \mathcal{L} unchanged.

Definition 5.8.

- (i) *A variational symmetry is a diffeomorphism $\Phi : E = \mathbb{R}^k \times Q \rightarrow E = \mathbb{R}^k \times Q$ verifying the following conditions:*
 - (a) *It is a fiber-preserving map for the bundle $\pi : E \rightarrow \mathbb{R}^k$; that is, Φ induces a diffeomorphism $\varphi : \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $\pi \circ \Phi = \varphi \circ \pi$*
 - (b) *If $\tilde{x} = \varphi(x)$ for each $x \in \mathbb{R}^k$*

$$\int_{\tilde{\Omega}} (L \circ j^1(\Phi \circ \phi \circ \varphi^{-1}))(\tilde{x}) d^k \tilde{x} = \int_{\Omega} (L \circ j^1 \phi)(x) d^k x$$

where $\tilde{\Omega} = \varphi(\Omega)$.

- (ii) *An infinitesimal variational symmetry is a vector field $X \in \mathfrak{X}(\mathbb{R}^k \times Q)$ whose local flows are variational symmetries.*

The following results can be seen in Theorem 4.12 and in Corollary 4.30 [34].

Theorem 5.9. *i) A vector field X on $\mathbb{R}^k \times Q$ is a variational symmetry if, and only if, $X^1(L) + L d_{T^*(0)} X_\alpha = 0$, where $X_\alpha = dx^\alpha(X)$.*

ii) If X is a variational symmetry then $\Theta_L^\alpha(X^1)$ defines a conservation law.

Example 5.10. We consider again the homogeneous isotropic 2-dimensional wave equation (5.7).

The rotation group $X = -x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3}$ is a variational symmetry, and then the corresponding conservation law $(\Theta_L^1(X^1), \Theta_L^2(X^1), \Theta_L^3(X^1))$ is given by the functions

$$\left(x^3 v_1 v_2 - x^2 v_1 v_3, -\frac{1}{2} x^3 u + x^2 v_2 v_3, -\frac{1}{2} x^2 u - v_3 v_2 x^3 \right)$$

where $u = (v_1)^2 + (v_2)^2 + (v_3)^2$.

Example 5.11. We consider again $Q = \mathbb{R}$, and let

$$0 = (1 + (\partial_2 \phi)^2) \partial_{11} \phi - 2 \partial_1 \phi \partial_2 \phi \partial_{12} \phi + (1 + (\partial_1 \phi)^2) \partial_{22} \phi$$

be the equation of minimal surfaces, which is the Euler-Lagrange equations for the Lagrangian $L : \mathbb{R}^2 \times T_2^1 \mathbb{R} \rightarrow \mathbb{R}$ defined by $L(x^1, x^2, q, v_1, v_2) = \sqrt{1 + (v_1)^2 + (v_2)^2}$.

The vector field $X = -q \frac{\partial}{\partial x^1} - q \frac{\partial}{\partial x^2} + (x^1 + x^2) \frac{\partial}{\partial q}$ is a variational symmetry, and then the corresponding conservation law $(\Theta_L^1(X^1), \Theta_L^2(X^1))$ is given by the functions

$$\left(\frac{-q(1 + (v_2)^2 - v_1 v_2) + (x^1 + x^2) v_1}{\sqrt{1 + (v_1)^2 + (v_2)^2}}, \frac{-q(1 + (v_1)^2 - v_1 v_2) + (x^1 + x^2) v_2}{\sqrt{1 + (v_1)^2 + (v_2)^2}} \right)$$

Now we describe some relationships between the above symmetries.

Theorem 5.12. *i) Let X be a π -vertical variational symmetry. Then the vector field $X \circ \pi_{1,0}$ along $\pi_{1,0}$ is a generalized symmetry.*

ii) The conservation law induced by X and $X \circ \pi_{1,0}$ coincide.

Proof. i) Since X is a π -vertical variational symmetry, then locally $X = X^i(x, q) \frac{\partial}{\partial q^i}$, and from Theorem 5.9 we know that $X^1(L) = 0$. From (3.8) we know that $(X \circ \pi_{1,0})^{(1)} = X^1 \circ \pi_{2,1}$. Then,

$$d_{(X \circ \pi_{1,0})^{(1)}} L(j_x^2 \phi) = (X \circ \pi_{1,0})^{(1)}(j_x^2 \phi)(L) = X^1(j_x^1 \phi)(L) = 0$$

and thus X is a generalized symmetry.

ii) It is a consequence of $\Theta_L^\alpha(X^1) = (\Theta_L^\alpha)^V(X \circ \pi_{1,0})$. \square

5.3. Noether symmetries. In the paper [29] we introduced the following definition:

Definition 5.13. *A vector field $Y \in \mathfrak{X}(\mathbb{R}^k \times T_k^1 Q)$ is an infinitesimal Noether symmetry if*

$$\mathcal{L}_Y \omega_L^\alpha = 0, \quad i_Y dx^\alpha = 0, \quad \mathcal{L}_Y E_L = 0.$$

Theorem 5.14. *Let X be a π -vertical vector field on $\mathbb{R}^k \times Q$ such that X^1 is a Noether symmetry, then $X \circ \pi_{1,0}$ is a generalized symmetry.*

Proof. Using (3.5), (4.10), then from the local expression $X = X^i \frac{\partial}{\partial q^i}$ and the condition $\mathcal{L}_{X^1} E_L = 0$, we deduce that

$$(X^i \circ \pi_{1,0}) \frac{\partial L}{\partial q^i} = v_\alpha^i X^1 \left(\frac{\partial L}{\partial v_\alpha^i} \right) \quad . \quad (5.8)$$

From the condition $\mathcal{L}_{X^1} \omega_L^\alpha = 0$ we obtain $d(\mathcal{L}_{X^1} \theta_L^\alpha) = 0$ and so there exist (locally defined) functions $F^\alpha: U \subset \mathbb{R}^k \times T_k^1 Q \rightarrow \mathbb{R}$ such that

$$\mathcal{L}_{X^1} \theta_L^\alpha = dF^\alpha \quad 1 \leq \alpha \leq k.$$

With these identities we obtain the following relations

$$\begin{aligned} \frac{\partial F^\alpha}{\partial x^\beta} &= \frac{\partial L}{\partial v_\alpha^i} \frac{\partial X^i}{\partial x^\beta} \circ \pi_{1,0} & \frac{\partial F^\alpha}{\partial q^j} &= \frac{\partial L}{\partial v_\alpha^i} \frac{\partial X^i}{\partial q^j} \circ \pi_{1,0} - X^1 \left(\frac{\partial L}{\partial v_\alpha^j} \right) \\ \frac{\partial F^\alpha}{\partial v_\beta^j} &= \frac{\partial L}{\partial v_\alpha^i} \frac{\partial X^i}{\partial v_\beta^j} \circ \pi_{1,0} = 0 \end{aligned} \quad (5.9)$$

From (3.8), (5.8) and (5.9), we deduce that

$$\begin{aligned} d_{(X \circ \pi_{1,0})^{(1)}} L(j_x^2 \phi) &= X^i(\phi(x)) \frac{\partial L}{\partial q^i} \Big|_{j_x^1 \phi} + \left(\frac{\partial X^i}{\partial x^\alpha} \Big|_{\phi(x)} + v_\alpha^j(j_x^1 \phi) \frac{\partial X^i}{\partial q^j} \Big|_{j_x^1 \phi} \right) \frac{\partial L}{\partial v_\alpha^i} \Big|_{j_x^1 \phi} \\ &= X^i(\phi(x)) \frac{\partial L}{\partial q^i} \Big|_{j_x^1 \phi} + \frac{\partial F^\alpha}{\partial x^\alpha} \Big|_{j_x^1 \phi} + v_\alpha^j(j_x^1 \phi) \left(\frac{\partial F^\alpha}{\partial q^j} \Big|_{j_x^1 \phi} + X^1(j_x^1 \phi) \left(\frac{\partial L}{\partial v_\alpha^j} \right) \right) \\ &= \frac{\partial F^\alpha}{\partial x^\alpha} \Big|_{j_x^1 \phi} + v_\alpha^j(j_x^1 \phi) \frac{\partial F^\alpha}{\partial q^j} \Big|_{j_x^1 \phi} = d_{T_\alpha^{(1)}} F^\alpha(j_x^2 \phi) \end{aligned}$$

for any $j_x^2 \phi$.

This proves that $X \circ \pi_{1,0}$ is a generalized symmetry. \square

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